XXVII - Fragmente über die Grenzfälle der elliptischen Modulfunctionen - Transcription de Dedekind

Auteurs : Richard Dedekind ; Bernhard Riemann

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- Bernhard Riemann
- Richard Dedekind

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Les relations du document

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XXVII - Fragmente über die Grenzfälle der elliptischen Modulfunctionen - Originaux de Riemann *suivant ce document*

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XXVII

Fragmente über die Grenzfälle der elliptischen Modulfunctionen.

([illegible]
[illegible])

I. Additamentum ad §um 20.

[Fundamenta nova theoriae functionum ellipticarum]

Formulae in hoc §o propositae in eo casu, ubi modulus ipsius q unitatem aequat, consideratione satis dignae videntur, quippe quae functiones unius variabilis pro quovis argumento valde discontinuas praebeant ~~(exhibent)~~

Series quidem propositae magna ex parte pro modulo ipsius q unitati aequali non convergunt, sed integrando series convergentes inde derivari possunt; itaque jam nunc integralia formularum 1–9 proponimus ~~(exhibemus)~~

$$48)\ \int_0 (\log k - \log 4\sqrt{q})\frac{dq}{q} = -4\log(1+q) + \frac{4}{4}\log(1+q^2) - \frac{4}{9}\log(1+q^3) + \frac{4}{16}\log(1+q^4) - \ldots$$

$$49)\ \int_0 -\log k' \frac{dq}{q} = 4\log\frac{1+q}{1-q} + \frac{4}{9}\log\frac{1+q^3}{1-q^3} + \frac{4}{25}\log\frac{1+q^5}{1-q^5}$$

$$50)\ \int_0 \log\frac{2K}{\pi}\frac{dq}{q} = 4\log(1+q) + \frac{4}{9}\log(1+q^3) + \frac{4}{25}\log(1+q^5)$$

$$51)\ \int_0 \left(\frac{2K}{\pi} - 1\right)\frac{dq}{q} = -4\log(1-q) + \frac{4}{3}\log(1-q^3) - \frac{4}{5}\log(1-q^5)$$
$$= +2i\log\frac{1-qi}{1+qi} + \frac{2i}{3}\log\frac{1-q^3 i}{1+q^3 i} + \frac{2i}{5}\log\frac{1-q^5 i}{1+q^5 i}$$

$$52)\ \int_0 \frac{2kK}{\pi}\frac{dq}{q} = 4\log\frac{1+\sqrt{q}}{1-\sqrt{q}} - \frac{4}{3}\log\frac{1+\sqrt{q^3}}{1-\sqrt{q^3}} + \frac{4}{5}\log\frac{1+\sqrt{q^5}}{1-\sqrt{q^5}}$$
$$= 4i\log\frac{1-\sqrt{q}\,i}{1+\sqrt{q}\,i} + \frac{4i}{3}\log\frac{1-\sqrt{q^3}\,i}{1+\sqrt{q^3}\,i} + \frac{4i}{5}\log\frac{1-\sqrt{q^5}\,i}{1+\sqrt{q^5}\,i}$$

$$53)\ \int_0 \left(\frac{2k'K}{\pi} - 1\right)\frac{dq}{q} = -4\log(1+q) + \frac{4}{3}\log(1+q^3) - \frac{4}{5}\log(1+q^5)$$
$$= -2i\log\frac{1-qi}{1+qi} + \frac{2i}{3}\log\frac{1-q^3 i}{1+q^3 i} - \frac{2i}{5}\log\frac{1-q^5 i}{1+q^5 i}$$

$$54)\ \int_0 \left(\frac{2kk'K}{\pi} - 1\right)\frac{dq}{q} = -\frac{4}{2}\log(1+q^2) + \frac{4}{6}\log(1+q^6) - \frac{4}{10}\log(1+q^{10}) + \frac{4}{14}\log(1+q^{14})$$
$$= -\frac{2i}{2}\log\frac{1-q^2 i}{1+q^2 i} + \frac{2i}{4}\log\frac{1-q^4 i}{1+q^4 i} - \frac{2i}{6}\log\frac{1-q^6 i}{1+q^6 i} + \frac{2i}{8}\log\frac{1-q^8 i}{1+q^8 i}$$

ubi logarithmos illos sumendos esse manifestum est, ut evanescant posito $q = 0$.

Functiones easdem ad dignitates ipsius q evolutas adhibitis Cl. Jacobi denotationibus hoc modo repraesentamus

55) $\int_0 (\log k - \log 4\sqrt{q})\frac{dq}{q} = -4\sum \frac{\psi(p)}{p^2}\left(q^p - \frac{3q^{2p}}{4} - \frac{1}{16}q^{4p} - \frac{1}{64}q^{8p} - \frac{1}{256}q^{16p} \dots\right)$

56) $\int_0 -\log k' \frac{dq}{q} = 8\sum \frac{\psi(p)}{p^2}\, q^p$

57) $\int_0 \log\frac{2K}{\pi}\,\frac{dq}{q} = 4\sum \frac{\psi(p)}{p^2}\left(q^p - \frac{1}{2}q^{2p} - \frac{1}{4}q^{4p} - \frac{1}{8}q^{8p} - \frac{1}{16}q^{16p} \dots\right)$

58) $\int_0 \left(\frac{2K}{\pi} - 1\right)\frac{dq}{q} = 4\sum \frac{\psi(n)\, q^{2^l(4m-1)^2 n}}{2^l(4m-1)^2 n}$

59) $\int_0 \left(\frac{2kK}{\pi}\right)\frac{dq}{q} = 8\sum \frac{\psi(n)\, q^{(4m-1)^2 n}}{(4m-1)^2 n}$

60) $\int_0 \left(\frac{2k'K}{\pi} - 1\right)\frac{dq}{q} = -4\sum \frac{\psi(n)\, q^{(4m-1)^2 n}}{(4m-1)^2 n} + 4\sum \frac{\psi(n)\, q^{2^{l+1}(4m-1)^2 n}}{2^{l+1}(4m-1)^2 n}$

61) $\int_0 \left(\frac{2\sqrt{k'}K}{\pi} - 1\right)\frac{dq}{q} = -4\sum \frac{\psi(n)\, q^{2(4m-1)^2 n}}{2(4m-1)^2 n} + 4\sum \frac{\psi(n)\, q^{2^{l+2}(4m-1)^2 n}}{2^{l+2}(4m-1)^2 n}$

Accuratiori ~~sint~~ functionum [...] propositarum disquisitioni tanquam lemma antemittimus theorema sequens generale:

Si series

$a_0 + a_1 + a_2$

ea quo scripsimus ordine summata summam habet convergentem, functio ipsius r per ~~seriem~~ (hac serie)

$a_0 + a_1 r + a_2 r^2$.

expressa (~~definita~~), convergente r versus limitem 1, convergit versus valorem eundem.

Hinc facile deducitur

Si functio $f(q)$ complexae quantitatis q pro modulis ipsius q unitate minoribus (~~crescente modulo ipsius q infra unitatem~~) exhibeatur per seriem

$a_0 + a_1 q + a_2 q^2 \dots$

hanc seriem pro valore q_0 cujus modulus sit unitas, si habeat summam, exprimere valorem eum, quem functio $f(q)$ ~~assumat~~ (nanciscatur) convergente q versus q_0 ita, ut modulus tantum mutetur, i. e. secundum notam repraesentationem geome tionem, appropinquante puncto, per quod quantitas q repraesentatur, in linea ad limitem spatii, pro quo ~~(ubi)~~ functio est data, normali.

Quamobrem hos tantum valores functionum propositarum ~~(in series evolutarum)~~ hic respiciemus, etiamsi ~~(quamquam)~~ evolutiones 48–54 latius pateant.

Sit brevitatis gratia $((x))$ aut absolute minima quantitatum a quantitate ~~(ubi)~~ x numero integro distantium, aut, si x ex numero integro et fractione $\frac{1}{2}$ composita est, $= 0$, porro $E(x)$ numerus integer maximus non major quam x: obtinemus e 48, attribuendo ipsi q valorem $q_0 = e^{xi}$

$$(2.)\int_0^{e^{xi}} (\log k - \log 4\sqrt{q})\frac{dq}{q} = -2\log 2\cos\frac{x}{2}^2 + \frac{2}{4}\log 2\cos\frac{2x}{2}^2 - \frac{2}{9}\log 2\cos\frac{3x}{2}^2 + \frac{2}{16}\log 2\cos\frac{4x}{2}^2$$

$$-4\pi i\left(\left(\frac{x}{2\pi}\right)\right) + \frac{4\pi i}{4}\left(\left(\frac{2x}{2\pi}\right)\right) - \frac{4\pi i}{9}\left(\left(\frac{3x}{2\pi}\right)\right) + \frac{4\pi i}{16}\left(\left(\frac{4x}{2\pi}\right)\right)$$

$$= 2\sum \frac{(-1)^n \log 2\cos\frac{nx}{2}^2}{nn} \quad \left[+ 4\pi i \sum \frac{(-1)^n}{nn}\left(\left(\frac{nx}{2\pi}\right)\right)\right]$$

Pars imaginaria hujus seriei convergit, quicunque est valor ipsius x, pars realis, si $\frac{x}{\pi}$ est numerus ~~(quantitas)~~ rationalis; non convergit, sin minus, designando literis m, n numeros integros inter se primos, et ponendo $\frac{x}{2\pi} = \frac{m}{n}$ ita exhiberi potest

1° si n est impar, aequalis fit,

$$\frac{\pi^2}{n^2}\sum_{1,n-1}^{s} \frac{(-1)^s \cos\frac{s\pi}{n}}{\sin\frac{s\pi}{n}} \log 2\cos\frac{sm\pi}{n}^2 - \frac{\pi^2}{6n^2}\log 2$$

2° si n est par, designante p numerum imparem

$$= \frac{\pi^2}{n^2}\sum_{1,n-1}^{s} \frac{2(-1)^s \log 2\cos\frac{sm\pi}{n}^2}{\sin s\frac{\pi}{n}} + \frac{\pi^2}{6n^2}\log 2 + \frac{3\pi^2}{4n^2}(-1)^{\frac{n}{2}}\left(\log\frac{q_0-q}{q_0+q} + \log n + \frac{8}{\pi^2}\sum\frac{\log p}{p^2}\right)$$

quae formula manifesto ita est intelligenda, functionem propositam, subtracta functione $\frac{2\pi}{n^2}(-1)^{\frac{n}{2}}\log\frac{q_0-q}{q_0+q}$, si convergat q modo cuique stabilite versus limitem q_0, convergere sursus limitem finitum, et usque valorem assignat.

Reinde obtinetur

$$(63)\int_0^{e^{xi}} -\log k' \frac{dq}{q} = -2\log tg\frac{x}{2}^2 - \frac{2}{9}\log tg\left(\frac{3x}{2}\right)^2 - \frac{2}{25}\log tg\left(\frac{5x}{2}\right)^2 - \cdots$$

$$+4\pi i\left(\left(\frac{x}{2\pi}\right)-\left(\frac{x}{2\pi}+\frac{1}{2}\right)\right)+\frac{4\pi i}{9}\left(\left(\frac{3x}{2\pi}\right)-\left(\frac{3x}{2\pi}+\frac{1}{2}\right)\right)+\frac{4\pi i}{25}\left(\left(\frac{5x}{2\pi}\right)-\left(\frac{5x}{2\pi}+\frac{1}{2}\right)\right)\cdots$$

$$= -\sum_{-\infty,\infty}\frac{\log tg\left(\frac{px}{2}\right)^2}{p^2} + \left[4\pi i\sum_{1}^{\infty}\frac{1}{p^2}\left(\left(\frac{px}{2\pi}\right)-\left(\frac{px}{2\pi}+\frac{1}{2}\right)\right)\right]$$

$$(64)\int_0^{e^{xi}}\log\frac{2K}{\pi}\frac{dq}{q} = 2\log 4\cos\left(\frac{x}{2}\right)^2+\frac{2}{9}\log 4\cos\frac{3x}{2}^2\cdots = \sum_{-\infty,\infty}\frac{\log 4\cos\frac{px}{2}^2}{p^2}\left[+4\pi i\sum_{1}^{\infty}\frac{1}{p^2}\left(\frac{px}{2\pi}\right)\right]$$

$$+4\pi i\left(\frac{x}{2\pi}\right)+\frac{4\pi i}{9}\left(\frac{3x}{2\pi}\right)+\frac{4\pi i}{25}\left(\frac{5x}{2\pi}\right)+\cdots$$

$$(65)\int_0^{e^{xi}}\left(\frac{2K}{\pi}-1\right)\frac{dq}{q} = -2\log 4\sin\frac{x}{2}^2+\frac{2}{3}\log 4\sin\frac{3x}{2}^2-\frac{2}{5}\log 4\sin\left(\frac{5x}{2}\right)^2\cdots$$

$$-4\pi i\left(\frac{x}{2\pi}+\frac{1}{2}\right)+\frac{4\pi i}{3}\left(\frac{3x}{2\pi}+\frac{1}{2}\right)\cdots$$

$$= i\log tg\left(\frac{x+\pi}{4}\right)^2+\frac{i}{2}\log tg\left(\frac{3x+\pi}{4}\right)^2+\frac{i}{3}\log tg\left(\frac{5x+\pi}{4}\right)^2\cdots$$

$$+2\pi\left(\left(\frac{x}{2\pi}+\frac{1}{4}\right)-\left(\frac{x}{2\pi}+\frac{3}{4}\right)\right)+\frac{2\pi}{3}\left(\left(\frac{3x}{2\pi}+\frac{1}{4}\right)-\left(\frac{3x}{2\pi}+\frac{3}{4}\right)\right)+\frac{2\pi}{5}\left(\left(\frac{5x}{2\pi}+\frac{1}{4}\right)-\left(\frac{5x}{2\pi}+\frac{3}{4}\right)\right)\cdots$$

$$(66)\int_0^{e^{xi}}\frac{2kK}{\pi}\frac{dq}{q} = -2\log tg\frac{x}{4}^2+\frac{2}{3}\log tg\frac{3x}{4}^2-\frac{2}{5}\log tg\frac{5x}{4}^2\cdots$$

$$+4\pi i\left(\left(\frac{x}{4\pi}\right)-\left(\frac{x}{4\pi}+\frac{1}{2}\right)\right)-\frac{4\pi i}{3}\left(\left(\frac{3x}{4\pi}\right)-\left(\frac{3x}{4\pi}+\frac{1}{2}\right)\right)\cdots$$

$$= 2i\log tg\left(\frac{x+\pi}{4}\right)^2+\frac{2i}{3}\log tg\frac{3x+\pi}{4}^2+\frac{2i}{5}\log tg\left(\frac{5x+\pi}{4}\right)^2+\cdots$$

$$+4\pi\left(\left(\frac{x}{4\pi}+\frac{1}{4}\right)-\left(\frac{x}{4\pi}+\frac{3}{4}\right)\right)+\frac{4\pi}{3}\left(\left(\frac{3x}{4\pi}+\frac{1}{4}\right)-\left(\frac{3x}{4\pi}+\frac{3}{4}\right)\right)\cdots$$

$$(67)\int_0^{e^{xi}}\left(\frac{2k'K}{\pi}-1\right)\frac{dq}{q} = -2\log 4\cos\frac{x}{2}^2+\frac{2}{3}\log 4\cos\frac{3x}{2}^2-\frac{2}{5}\log 4\cos\frac{5x}{2}^2\cdots$$

$$-4\pi i\left(\frac{x}{2\pi}\right)+\frac{4\pi i}{3}\left(\frac{3x}{2\pi}\right)-\frac{4\pi i}{5}\left(\frac{5x}{2\pi}\right)\cdots$$

$$= -i\log tg\left(\frac{x+\pi}{4}\right)^2+\frac{i}{2}\log tg\left(\frac{3x+\pi}{4}\right)^2-\frac{i}{3}\log tg\left(\frac{5x+\pi}{4}\right)^2\cdots$$

$$-2\pi\left(\left(\frac{x}{2\pi}+\frac{1}{4}\right)-\left(\frac{x}{2\pi}+\frac{3}{4}\right)\right)+\frac{2\pi}{3}\left(\left(\frac{3x}{2\pi}+\frac{1}{4}\right)-\left(\frac{3x}{2\pi}+\frac{3}{4}\right)\right)\cdots$$

$$(68)\int_0^{e^{xi}}\left(\frac{2\sqrt{k'}K}{\pi}-1\right)\frac{dq}{q} = -\log 4\cos x^2+\frac{1}{3}\log 4\cos 3x^2-\frac{1}{5}\log 4\cos 5x^2+\cdots$$

$$-2\pi i\left(\frac{x}{\pi}\right)+\frac{2\pi i}{3}\left(\frac{3x}{\pi}\right)-\frac{2\pi i}{5}\left(\frac{5x}{\pi}\right)+\cdots$$

$$= -\frac{i}{2}\log tg\left(x+\frac{\pi}{4}\right)^2+\frac{i}{4}\log tg\left(3x+\frac{\pi}{4}\right)^2-\frac{i}{6}\log tg\left(5x+\frac{\pi}{4}\right)^2+\cdots$$

$$-\pi\left(\left(\frac{x}{\pi}+\frac{1}{4}\right)-\left(\frac{x}{\pi}+\frac{3}{4}\right)\right)+\frac{\pi}{3}\left(\left(\frac{3x}{\pi}+\frac{1}{4}\right)-\left(\frac{3x}{\pi}+\frac{3}{4}\right)\right)-\frac{\pi}{5}\left(\left(\frac{5x}{\pi}+\frac{1}{4}\right)-\left(\frac{5x}{\pi}+\frac{3}{4}\right)\right)\cdots$$

Posito $x = \frac{m}{n}\pi$ fit pars imaginaria formulae 65

1° si n est numerus par

$$= \sum_{0,\infty}^{s} -4\pi i \sum_{1,n-1}^{\mu} (-1)^{\frac{\mu-1}{2}} \frac{1}{\mu+ns} \left(\frac{\mu m}{n}+\frac{1}{2}\right)(-1)^{\frac{m}{2}s}$$

2° si n est numerus impar

$$= \sum_{0,\infty}^{s} -4\pi i \sum_{1,2n-1}^{\mu} (-1)^{\frac{\mu-1}{2}} \frac{1}{\mu+2ns} \left(\frac{\mu m}{n}+\frac{1}{2}\right)(-1)^{s}$$

quam patet habere valorem finitum, nisi n est $\equiv 0$ mod 4.

Convergentia summae

$$a_0 + a_1 + a_2 + a_3 \ldots\ldots$$

postulat, ut data quantitate quamvis parva ε assignari possit terminus a_n, a quo summa usque ad terminum quemvis a_m extensa nanciscatur valorem absolutum ipso ε minorem. Jam posito brevitatis gratia

$$\varepsilon_{n+1} = a_{n+1}$$
$$\varepsilon_{n+2} = a_{n+1} + a_{n+2}$$
$$\varepsilon_{n+3} = a_{n+1} + a_{n+2} + a_{n+3}$$

functio

$$f(r) = a_0 + a_1 r + a_2 r^2 + \ldots + \ldots$$

facile sub hac forma exhibetur

$$= a_0 + a_1 r + a_2 r^2 \ldots a_n r^n + \varepsilon_{n+1} r^{n+1} + (\varepsilon_{n+2} - \varepsilon_{n+1}) r^{n+2} + (\varepsilon_{n+3} - \varepsilon_{n+2}) r^{n+3}$$

$$= a_0 \ldots\ldots a_n r^n + \varepsilon_{n+1}(r^{n+1} - r^{n+2}) + \varepsilon_{n+2}(r^{n+2} - r^{n+3}) + \ldots$$

Unde patet convergente r versus limitem 1 functionem $f(r)$ tandem quavis quantitate minus a valore seriei

$$a_0 + a_1 + a_2 \ldots$$

distare. Summa terminorum altioris gradus quam n, quum sint $\varepsilon_{n+1}, \varepsilon_{n+2} \ldots$ ex hyp. omnes omisso ~~(abjecto)~~ signo $< \varepsilon$, differentiaeque $r^{n+1} - r^{n+2} \ldots$ omnes positivae, manifesto evadit quantitate absoluta

$< \varepsilon (r^{n+1} - r^{n+2}) + \varepsilon (r^{n+1} - r^{n+2})$.

$< \varepsilon r^{n+1}$

summa autem terminorum non altioris gradus quam n est functio algebraica ipsius r, quam constat [illegible] summae

$a_0 + a_1 + a_2 + \dots + a_n$

quantumvis ~~admoveri~~ (appropinquari posse; unde patet appropinquando r unitati differentiam functionis $f(r)$ a valore seriei

$a_0 + a_1 + \dots$

infra quantitatem quamvis datam descendere.

433/28

[appropinquando r unitati

Ex hoc theoremate, quod Cl° Abel tribuendum esse Cl^us Dirichlet ~~modo~~ (1852 Sept 14) quum antecedentia jam essent scripta monuit, facile deducitur ....

2) $x = \frac{m\pi}{n}$; n impar

$$-\log k' = -\frac{q+q_0}{q-q_0}\,\frac{\pi^2}{4n^2}\,q_0^{-n} + 8\int_0^1 \sum_{1,2n-1}^{s} \frac{x^{s-1}dx}{1-x^{2n}}\,\frac{x^{ms}}{1-x^{2ms}}$$

$$= A + 8\int_0^1 \sum_{1,2n}^{t} \frac{x^t dx}{1-x^{t}x}\,\frac{1}{2n}\,\frac{1}{2n}\sum_{1,2n-1}^{s}\sum_{0,n-1}^{r}\left(\frac{s-\frac{n+1}{2}}{n}\right)x^{ms(2r+1)}x^{-ts}$$

1) $t = m(2r+1) \mod 2n$

2) $t = m(2r+1)+n$

$$= A + 8\sum_{0,n-1}\log\left(1-x^{m(2r+1)}\right)\frac{1}{2n}\left(\frac{r-\frac{n+1}{2}}{n}\right)n - 8\sum\log\left(1-x^{m(2r+1)+n}\right)\frac{1}{2n}\,n\left(\frac{r-\frac{n+1}{2}}{n}\right)$$

$$= A + 8\sum_{1,\frac{n-1}{2}}\frac{1}{2}\left(\frac{s}{n}\right)\left(\log\left(1-x^{2ms+mn}\right) - \log\left(1-x^{2ms+mn}\right)\right)$$

$$- 4\sum\left(\frac{s}{n}\right)\left(\log\left(1-x^{2ms+(m+1)n}\right) - \log\left(1-x^{2ms+(m+1)n}\right)\right)$$

$$= A + 8\pi\sum_{1,\frac{n-1}{2}}\left(\frac{s}{n}\right)\left(\frac{2ms+(m+1)n}{2n}\right) - \left(\frac{2ms+mn}{2n}\right)$$

$$= A + 4\pi i\sum\left(\frac{s}{n}\right)\left(\cdots\cdots\right)$$

$$= A + 4\pi i\sum\left(\frac{\mu s}{n}\right)\left(\left(\frac{2s+(m+1)n}{2n}\right) - \left(\frac{2s+mn}{2n}\right)\right), \quad m\mu \equiv 1 \mod n$$

$$= A + 4\pi i(-1)^{m+1}\sum_{1,\frac{n-1}{2}}\left(\frac{\mu s}{n}\right)$$

$$= (-1)^{m+1}\left(\frac{\pi^2}{4n^2}\,\frac{q+q_0}{q-q_0} + \pi i\left(\frac{n^2-1}{n}\mu - 4\sum_{1,\frac{n-1}{2}}\varepsilon\left(\frac{\mu s}{n}+\frac{1}{2}\right)\right)\right)$$

$$\log\frac{2K}{\pi} = 4\sum\frac{q^t}{t(1+q^{t})} - \log\left(\frac{q_0+q}{q_0-q}\right) + 4\sum\frac{1}{t}\left(\frac{q^t}{1+q^t} - \frac{1}{2}\,\frac{q^t}{q_0^{\,t}}\right)$$

$$= \log\frac{q_0+q}{q_0-q} + 2i\sum\frac{1}{t}\,\mathrm{tg}\,\frac{t\alpha}{2}.$$

1) $z = \frac{2m}{n}\pi$, n impar.

$$x = e^{\frac{2\pi}{2n}i}, \quad \frac{1}{1+\tau x^{2m}} = \sum_{0,n-1}\frac{(-1)^{s}\tau^{s}x^{2sm}}{1+\tau^{n}}$$

$$\log\frac{2K}{\pi} = \log\frac{q_0+q}{q_0-q} + 2\sum\sum_{1,m-1}^{s}\frac{1}{2nr+s}\,\frac{x^{2ms}-1}{x^{2ms}+1}$$

$$= \log\frac{q_0+q}{q_0-q} + 2\int_0^1\sum_{1,2n}^{t}\frac{x^t dx}{1-x^t x}\,\frac{1}{2n}\,x^{-ts}\sum_{1,n-1}^{s}(-1)^s x^{2s m}$$

$$= \log\frac{q_0+q}{q_0-q} + 2\sum_{1,n-1}\log\left(1-x^{2rm}\right)(-1)^r\frac{1}{2n}\,n - 2\sum_{1,n-1}\log\left(1-x^{2rm+n}\right)(-1)^r\frac{1}{2n}\,n$$

$$= A + \frac{1}{2}\sum\left(\frac{rm}{n}+\frac{1}{2}\right)(-1)^r 2\pi i - \frac{1}{2}\sum\left(\frac{rm}{n}\right)(-1)^r 2\pi i$$

$$= \log\frac{q_0+q}{q_0-q} + 2\pi i\sum_{1,\frac{n-1}{2}}^{s}\left(\left(s\frac{2m}{n}+\frac{1}{2}\right) - \left(s\frac{2m}{n}\right)\right)$$

---

2) $x = \frac{m}{n}\pi$, n ungerade, m ungerade, $x = e^{\frac{2\pi i}{4n}}$

$$\log\frac{2K}{\pi} = \frac{q+q_0}{q-q_0}\,\frac{\pi^2}{4n^2} + \log\frac{q_0+q}{q_0-q} + 2\sum\sum_{1,4n-1}^{s}\frac{1}{4nt+s}\,\frac{x^{2ms}-1}{x^{2ms}+1}$$

$$= A + 2\int_0^1 \sum_{1,4n}^{t}\frac{x^t\,dx}{1-x^{4n}x}\cdot\frac{1}{4n} - \frac{1}{2n}\sum_{1,4n-1}^{s}\sum_{\sigma,2n-1}^{\sigma}(-1)^\sigma\sigma\, x^{2s\sigma m}\left(x^{2ms}-1\right)x^{-ts}$$

~~1) $t \equiv 2ms$ mod $4n$~~

~~2) t~~

$$= A + 2\int_0^1 \sum_{1,4n}^{t}\frac{x^t\,dx}{1-x^{4n}x}\cdot\frac{1}{4n}\cdot 2\sum_{1,4n-1}^{s}\sum_{1,2n-1}^{\sigma}(-1)^\sigma\left(\frac{\sigma-n}{2n}\right)x^{2ms\sigma}x^{-ts}$$

1) $t \equiv 2m\sigma$ mod. $4n$

2) $t \equiv 2m\sigma + 2n$

$$= A - 2\sum_{1,2n-1}\log\left(1-x^{2m\sigma}\right)\frac{1}{4n}(-1)^\sigma\left(\frac{\sigma-n}{2n}\right)4n + 2\sum\log\left(1-x^{2m\sigma+2n}\right)(-1)^\sigma\left(\frac{\sigma-n}{2n}\right)$$

$$= A - 2\pi i\sum_{1,2n-1}(-1)^\sigma\left(\left(\left(\frac{m\sigma+n}{2n}\right)\right)-\left(\left(\frac{m\sigma}{2n}\right)\right)\right)\left(\frac{\sigma-n}{2n}\right)$$

$$= A - 2\pi i\sum_{1,2n-1}(-1)^\sigma\left(\left(\left(\frac{\sigma+n}{2n}\right)\right)-\left(\left(\frac{\sigma}{2n}\right)\right)\right)\left(\frac{\mu\sigma-n}{2n}\right)\ ,\quad m\mu \equiv 1 \text{ mod. } 2n$$

$$= A + 2\pi i\sum_{1,n-1}(-1)^\sigma\left(\frac{\mu\sigma-n}{2n}\right)$$

---

3) $x = \frac{m}{2n}\pi$, m ungerade.

$$\log\frac{2K}{\pi} = \log\frac{q_0+q}{q_0-q} + 2\sum\sum_{1,4n-1}^{s}\frac{1}{4nt+s}\,\frac{x^{ms}-1}{x^{ms}+1}$$

$$= A + 2\int_0^1\sum_{1,4n}^{t}\frac{x^t\,dx}{1-x^{4n}x}\cdot\frac{1}{4n}\cdot 2\sum_{1,4n-1}^{s}\sum_{1,4n-1}^{\sigma}(-1)^\sigma\left(\frac{\sigma-2n}{4n}\right)x^{ms\sigma}x^{-ts}$$

$$= A + 2\pi i\sum_{1,4n-1}(-1)^\sigma\left(\frac{\mu\sigma-2n}{4n}\right)\ ,\quad m\mu \equiv 1 \text{ mod. } 4n$$

~~[illegible]~~

Fragment II

$$\log k = \log 4\sqrt{q} + \sum (-1)^n \frac{4}{n} \frac{q^n}{1+q^n} \; ; \; q = e^{xi}$$ ~~[illegible]~~

1) $x = \frac{2m}{n}\pi$, n ungerade. ~~[illegible]~~

$$\lg 1 = i\left(\frac{\pi}{2} + \sum (-1)^s \frac{2}{s} \operatorname{tg} s\frac{x}{2}\right)$$

$$= i\left(\frac{\pi}{2} + \sum_{0,\infty}^{t} \sum_{1,2n}^{s} (-1)^s \frac{2}{2nt+s} \operatorname{tg}\frac{sm}{n}\pi\right)$$

$$= i\frac{\pi}{2} + 2i\int_0^1 \sum_{1,2n}^{s} (-1)^s \operatorname{tg}\frac{sm}{n}\pi \frac{x^{s-1}dx}{1-x^{2n}}$$

$$= i\frac{\pi}{2} + 2\int_0^1 \sum_{1,2n}^{s} (-1)^s \frac{\alpha^{2sm}-1}{\alpha^{2sm}+1} \cdot \frac{1}{2n} \sum_{1,2n}^{t} \frac{\alpha^{-ts}\alpha^t dx}{1-\alpha^t x}, \quad \alpha = e^{\frac{2\pi i}{2n}}$$

$$= i\frac{\pi}{2} + \frac{1}{2n}\int_0^1 \sum_{1,2n}^{t} \frac{\alpha^t dx}{1-\alpha^t x} \cdot 2\sum_{1,n-1} \sum_{1,2n}^{s} (-1)^{s+\sigma-1} \alpha^{s(2m\sigma-t)}$$

$$\frac{1}{1+\tau\alpha^{2sm}} = \sum \frac{(-1)^\sigma \alpha^{2s\sigma m} \tau^\sigma}{1-\tau^{2n}} = -\frac{1}{2n} \sum_{0,2n-1} (-1)^\sigma \sigma \alpha^{2s\sigma m}$$

$$= \frac{1}{2} \sum_{0,n-1} (-1)^\sigma \alpha^{2s\sigma m}$$

$$= i\frac{\pi}{2} + 2\sum_{1,n-1} \log\left(1-x^{n+2m\sigma}\right)(-1)^\sigma$$

$$= i\frac{\pi}{2} + \sum_{1,n-1} \log x^{2m\sigma} (-1)^\sigma$$

$$= i\frac{\pi}{2} + 2\pi i\left(\sum_{1,n-1} \frac{2m\sigma}{2n}(-1)^\sigma - \sum_{1,n-1} (-1)^\sigma \mathfrak{E}\left(\frac{2m\sigma}{2n} + \frac{1}{2}\right)\right)$$

---

2) $x = \frac{m}{n}\pi$, m, n ungerade.

$$\log k = -\frac{q+q_0}{q-q_0} \cdot \frac{2}{\lg 2} \sum_{1,\infty} \frac{1}{s^2} - \frac{1}{n}\lg\frac{1+q^n}{1-q^n} + \frac{\pi}{2}i + 2\int_0^1 \sum_{1,4n-1}^{s} (-1)^s \frac{x^{s-1}dx}{1-x^{4n}} \cdot \frac{x^{2ms}-1}{x^{2ms}+1}$$

$$= \mathfrak{A} + \frac{\pi}{2}i + 2\int_0^1 \sum_{1,4n}^{t} \frac{x^t dx}{1-x^t x} \cdot \frac{1}{4n} \cdot \frac{1}{2n} \sum_{1,4n-1}^{s} \sum_{0,2n-1}^{\sigma} (-1)^{s+\sigma} \sigma x^{2s\sigma m} (\alpha^{2ms}-1) x^{-st}$$

$$= \mathfrak{A} + \frac{\pi}{2}i + 2 \cdot 2n \sum_{1,n-1} \frac{s}{n}(-1)^s \left(\frac{ms-n}{2n} - \mathfrak{E}\frac{ms}{2n}\right), \quad m\mu \equiv 1 \pmod{2n}$$

$$= \mathfrak{A} + \pi i\left(\frac{m-\mu}{2} + \frac{\mu}{2n} + 2\sum_{1,n-1} \mathfrak{E}\frac{\mu s}{2n}(-1)^s - 2\sum_{1,n-1} \mathfrak{E}\frac{ms}{2n}(-1)^s\right)$$

---

2) $x = \frac{m}{2n}\pi$, m ungerade

$$\log h = \frac{q+q_0}{q-q_0}\;\frac{3}{8n^2}\sum\frac{1}{s^2} + \frac{1}{2n}\log\left(\frac{1+q^{2n}}{1-q^{2n}}\right)$$

$$+\frac{\pi}{2}i + \sum^{t}\sum_{1,2n-1}^{s}(-1)^{s}\frac{2}{8nt+s}\,tg\,s\frac{m}{4n}\pi$$

$$= A + \frac{\pi}{2}i + 2\int_0^1\sum_{1,2n-1}^{s}\frac{x^{s-1}dx}{1-x^{8n}}\;\frac{x^{2ms}-1}{x^{2ms}+1}(-1)^{s}$$

$$= A + \frac{\pi}{2}i + 2\int_0^1\sum_{1,8n}^{t}\frac{x^{t}dx}{1-x^{t}x}\;\frac{1}{8n} - \frac{1}{4n}\sum_{1,2n-1}^{s}\sum_{0,4n-1}^{\sigma}(-1)^{s+\sigma}\sigma x^{4\sigma s m}(x^{2ms}-1)x^{-st}$$

$$t = 2\sigma m + 4n \bmod 8n$$

$$= A + \frac{\pi}{2}i + 2\sum_{1,4n-1}^{\sigma}\log(1-x^{4n+2\sigma m})\frac{1}{8n}\;\frac{1}{4n}\left(8n((-1)^{\sigma}(\sigma-1)+(-1)^{\sigma}\sigma) + 8n(-1)^{\sigma}(4n-1)\right)$$

$$= A + \frac{\pi}{2}i + 2\sum_{-(n+1),2n-1}^{s}\log(1-x^{2sm})\frac{1}{2n}(-1)^{s}$$

$$= A + \frac{\pi}{2}i - 4\sum_{0,2n-1}^{s}\log(-x)^{2sm}\frac{1}{4n}(-1)^{s}$$

$$= A + \frac{\pi}{2}i - 4\sum_{0,2n-1}^{s}\left(\frac{sm}{4n}\right)\frac{1}{4n}(-1)^{s}2\pi i$$

(x) = absolut kleinster Rest von x.

$$-\log h' = 8\sum\frac{1}{t}\frac{q^{t}}{1-q^{2t}} = 4i\sum\frac{1}{t\sin tx}\;,\; q = e^{xi}.$$

(1)

1) $x = \frac{m}{2n}\pi$, m ungerade.

$$-\log h' = 4i\sum_{0,\infty}^{t}\sum_{1,4n-1}^{s}\frac{1}{4nt+s}\;\frac{1}{\sin\frac{sm\pi}{2n}}$$

$$= 8\int_0^1\sum^{s}\frac{x^{s-1}dx}{1-x^{4n}}\;\frac{x^{sm}}{1-x^{2ms}}$$

$$= 8\int_0^1\sum_{1,4n}^{t}\frac{x^{t}dx}{1-x^{t}x}\;\frac{1}{4n} - \frac{1}{2n}\sum_{1,4n-1}^{s}\sum_{0,2n-1}^{\sigma}\sigma x^{ms(2\sigma+1)}x^{-ts}$$

$$\frac{1}{1-rx^{2ms}} = \sum_{0,2n-1}^{\sigma}\frac{r^{\sigma}x^{2ms\sigma}}{1-r^{2n}}$$

$$\frac{1}{1-x^{2ms}} = \frac{1}{2n}\sum_{0,2n-1}^{\sigma}\sigma x^{2ms\sigma}$$

$$t = m(2\sigma+1) \bmod 4n$$

$$= 8\sum_{0,2n-1}^{\sigma}\log(1-x^{m(2\sigma+1)})\;\frac{1}{4n}\cdot\frac{1}{2n}\cdots$$

$$= \pi i\left((m-2)n - 4\sum_{0,n-1}^{s}\mathcal{E}\left(\frac{m(2s+1)}{4n}\right)\right)$$
